

On future geodesic completeness for the Einstein-Vlasov system with hyperbolic symmetry

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Abstract

Spacetimes with collisionless matter evolving from data on a compact Cauchy surface with hyperbolic symmetry are shown to be timelike and null geodesically complete in the expanding direction, provided the data satisfy a certain size restriction.

1 Introduction

Consider the Einstein-Vlasov system with hyperbolic symmetry. For background information on this system which describes the evolution of a collisionless gas in general relativity we refer to [4], for the notion of hyperbolic symmetry we refer to [1]. As shown in [1] the corresponding spacetime manifold can be covered by areal coordinates in which the metric takes the form

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} d\theta^2 + t^2 (d\psi^2 + \sinh^2 \psi d\phi^2), \quad (1.1)$$

where μ and λ are functions of t and θ , periodic in θ with period 1, and the areal time coordinate t takes all values in $]R_0, \infty[$ for some $R_0 \geq 0$. The main motivation of the latter investigation was to provide a set-up in which the global properties of solutions of the Einstein equations with hyperbolic symmetry coupled to a matter field, in this case a collisionless gas, can be studied. In the present note we give sufficient conditions on initial data such that the spacetime manifold is timelike and null geodesically complete towards the future, i. e., in the expanding direction. In the contracting direction such a spacetime has a spacetime singularity, cf. [1] or [3]. In the spherically symmetric, asymptotically flat case of the Einstein-Vlasov system geodesic completeness was obtained in [4] for small data. No analogous result exists for cosmological models.

The geodesic equations for a metric of the form (1.1) imply that along geodesics the variables t , θ , p^0 , and

$$w := e^\lambda p^1, \quad L := t^4 ((p^2)^2 + \sinh^2 \psi (p^3)^2)$$

satisfy the following system of differential equations:

$$\frac{d\theta}{d\tau} = e^{-\lambda}w, \quad \frac{dw}{d\tau} = -\lambda_t p^0 w - e^{2\mu-\lambda} \mu_\theta (p^0)^2, \quad \frac{dL}{d\tau} = 0, \quad (1.2)$$

$$\frac{dt}{d\tau} = p^0, \quad \frac{dp^0}{d\tau} = -\mu_t (p^0)^2 - 2e^{-\lambda} \mu_\theta p^0 w - e^{-2\mu} \lambda_t w^2 - e^{-2\mu} t^{-3} L. \quad (1.3)$$

For a particle with rest mass m moving forward in time p^0 can be expressed by the remaining variables,

$$p^0 = e^{-\mu} \sqrt{m^2 + w^2 + L/t^2} > 0,$$

and the corresponding geodesic as well as the solution of (1.2), (1.3) can be reparameterized by coordinate time t . Consider an ensemble of such particles, all with rest mass equal to unity. Due to the symmetry their density on the mass shell

$$\{g^{\alpha\beta} p^\alpha p^\beta = -1, p^0 > 0\}$$

can be written as

$$f = f(t, \theta, w, L),$$

cf. [1]. In these variables the Einstein-Vlasov system takes the form

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+L/t^2}} \partial_\theta f - \left(\lambda_t w + e^{\mu-\lambda} \mu_\theta \sqrt{1+w^2+L/t^2} \right) \partial_w f = 0, \quad (1.4)$$

$$e^{-2\mu} (2t\lambda_t + 1) - 1 = 8\pi t^2 \rho, \quad (1.5)$$

$$e^{-2\mu} (2t\mu_t - 1) + 1 = 8\pi t^2 p, \quad (1.6)$$

$$\mu_\theta = -4\pi t e^{\mu+\lambda} j, \quad (1.7)$$

$$e^{-2\lambda} \left(\mu_{\theta\theta} + \mu_\theta (\mu_\theta - \lambda_\theta) \right) - e^{-2\mu} \left(\lambda_{tt} + (\lambda_t + 1/t)(\lambda_t - \mu_t) \right) = 4\pi q, \quad (1.8)$$

where

$$\rho(t, \theta) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+L/t^2} f(t, \theta, w, L) dL dw, \quad (1.9)$$

$$p(t, \theta) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1+w^2+L/t^2}} f(t, \theta, w, L) dL dw, \quad (1.10)$$

$$j(t, \theta) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, \theta, w, L) dL dw, \quad (1.11)$$

$$q(t, \theta) := \frac{\pi}{t^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{L}{\sqrt{1+w^2+L/t^2}} f(t, \theta, w, L) dL dw. \quad (1.12)$$

The characteristics of the Vlasov equation (1.4) along which f is constant are the solutions of

$$\dot{\theta} = \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+L/t^2}}, \quad \dot{w} = -\lambda_t w - e^{\mu-\lambda} \mu_\theta \sqrt{1+w^2+L/t^2}, \quad \dot{L} = 0 \quad (1.13)$$

where the dot denotes differentiation with respect to coordinate time t and the relation with (1.2), (1.3) is obvious.

Initial data are prescribed at some time $t = t_0 > 0$:

$$f(t_0, \theta, w, L) = \mathring{f}(\theta, w, L), \quad \lambda(t_0, \theta) = \mathring{\lambda}(\theta), \quad \mu(t_0, \theta) = \mathring{\mu}(\theta).$$

If these data are C^1 and periodic in θ with period 1, if \mathring{f} is compactly supported with respect to w and L , and if the data satisfy the constraint equation (1.7), then the corresponding solution exists for all future areal time $t \geq t_0$, cf. [1]. To save a little notation let $t_0 = 1$.

In the next section a bound on w along characteristics of (1.4) is established for restricted data. In Section 3 the lapse function e^μ is estimated, again for restricted data. In the last section both estimates are combined to obtain future geodesic completeness.

2 An estimate along characteristics

Let

$$\begin{aligned} w_0 &:= \sup \left\{ |w| \mid (r, w, L) \in \text{supp } \mathring{f} \right\} < \infty, \\ L_0 &:= \sup \left\{ L \mid (r, w, L) \in \text{supp } \mathring{f} \right\} < \infty; \end{aligned}$$

without loss of generality $w_0 > 0$, $L_0 > 0$. For $t \geq 1$ define

$$\begin{aligned} P_+(t) &:= \max \left\{ 0, \max \left\{ w \mid (r, w, L) \in \text{supp } f(t) \right\} \right\}, \\ P_-(t) &:= \min \left\{ 0, \min \left\{ w \mid (r, w, L) \in \text{supp } f(t) \right\} \right\}. \end{aligned}$$

We claim that if

$$4\pi^2(1 + L_0)L_0 \|\mathring{f}\|_\infty < 1 \tag{2.1}$$

then

$$P_+(t) \leq w_0 \sqrt{t}, \quad P_-(t) \geq -w_0 \sqrt{t}, \quad t \geq 1. \tag{2.2}$$

Assume the estimate on P_+ were false for some t . Define

$$t_0 := \sup \{ t \geq 1 \mid P_+(s) \leq w_0 \sqrt{s}, \quad 1 \leq s \leq t \}$$

so that $1 \leq t_0 < \infty$ and $P_+(t_0) = w_0 \sqrt{t_0} > 0$. Choose $\epsilon \in]0, 1[$ such that

$$4\pi^2(1 + L_0)L_0 \|\mathring{f}\|_\infty \leq (1 - \epsilon)^2.$$

By continuity, there exists some $t_1 > t_0$ such that the following holds:

$$(1 - \epsilon)P_+(s) > 0, \quad s \in [t_0, t_1],$$

and if for some characteristic curve $(\theta(s), w(s), L)$ in the support of f , that is, with $(\theta(1), w(1), L) \in \text{supp } \mathring{f}$, and for some $t \in]t_0, t_1]$ the estimate

$$(1 - \epsilon/2)P_+(t) \leq w(t) \leq P_+(t) \quad (2.3)$$

holds, then

$$(1 - \epsilon)P_+(s) \leq w(s) \leq P_+(s), \quad s \in [t_0, t]; \quad (2.4)$$

note that the estimates on w from above hold by definition of P_+ in any case.

Let $(\theta(s), w(s), L)$ be a characteristic in the support of f satisfying (2.3) for some $t \in]t_0, t_1]$ and thus (2.4) on $[t_0, t]$. Then on $[t_0, t]$,

$$\begin{aligned} \dot{w} &= \frac{4\pi^2}{s} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w} \sqrt{1 + w^2 + L/s^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{L}/s^2} \right) f d\tilde{L} d\tilde{w} \\ &\quad + \frac{1 - e^{2\mu}}{2s} w \\ &\leq \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{L_0} \frac{\tilde{w}^2(1 + w^2 + L/s^2) - w^2(1 + \tilde{w}^2 + \tilde{L}/s^2)}{\tilde{w} \sqrt{1 + w^2 + L/s^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{L}/s^2}} f d\tilde{L} d\tilde{w} \\ &\quad + \frac{1 - e^{2\mu}}{2s} w \\ &\leq \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{L_0} \frac{\tilde{w}(1 + L)}{w} f d\tilde{L} d\tilde{w} + \frac{1 - e^{2\mu}}{2s} w \\ &\leq 4\pi^2 L_0(1 + L_0) \|\mathring{f}\|_{\infty} \frac{e^{2\mu}}{2s} P_+^2(s) \frac{1}{w} + \frac{1 - e^{2\mu}}{2s} w \\ &\leq \frac{e^{2\mu}}{2s} ((1 - \epsilon)P_+(s))^2 \frac{1}{w} + \frac{1 - e^{2\mu}}{2s} w \leq \frac{e^{2\mu}}{2s} w + \frac{1 - e^{2\mu}}{2s} w = \frac{1}{2s} w. \end{aligned}$$

Thus

$$w(t) \leq w(t_0) \sqrt{t/t_0} \leq P_+(t_0) \sqrt{t/t_0} = w_0 \sqrt{t}$$

by assumption on t_0 . This estimate holds only for characteristics which satisfy (2.3), but this is sufficient to conclude that

$$P_+(t) \leq w_0 \sqrt{t}, \quad t \in [t_0, t_1],$$

in contradiction to the choice of t_0 . The estimate on P_+ is now established.

The analogous arguments for characteristics with $w < 0$ yield the assertion for P_- , and we have shown:

Proposition 2.1 *For any solution of the Einstein-Vlasov system with hyperbolic symmetry written in areal coordinates and with initial data as above,*

$$|w| \leq w_0 \sqrt{t}, \quad (\theta, w, L) \in \text{supp } f(t), \quad t \geq 1,$$

provided the data satisfy the size restriction

$$4\pi^2(1 + L_0)L_0 \|\mathring{f}\|_{\infty} < 1.$$

Remark: The estimate above is sufficient to show that all geodesics which correspond to characteristics in the support of f exist for all proper time in the expanding direction, but geodesic completeness requires the same to hold for timelike geodesics in general.

3 An estimate for the lapse function

We reconsider some of the estimates in [1] from the point of view of getting a better bound on e^μ for restricted data. First recall that by integration of the field equation (1.6) and since p is non-negative,

$$e^{2\mu(t,\theta)} \geq \frac{t}{e^{-2\mathring{\mu}(\theta)} - 1 + t} \geq \frac{t}{c_0 - 1 + t}, \quad t \geq 1, \quad (3.1)$$

where

$$c_0 := \max_{\theta \in [0,1]} e^{-2\mathring{\mu}(\theta)}.$$

Next we recall that by a lengthy computation,

$$\begin{aligned} \frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta &= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q - \frac{\rho+p}{2} (1 - e^{2\mu}) \right] d\theta \\ &\leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho - \frac{\rho+p}{2} \right] d\theta - \frac{1}{c_0 - 1 + t} \int_0^1 e^{\mu+\lambda} \frac{\rho+p}{2} d\theta \\ &\leq -\frac{2}{c_1 + t} \int_0^1 e^{\mu+\lambda} \rho d\theta \end{aligned}$$

where

$$c_1 := \max\{0, c_0 - 1\}.$$

Thus

$$\int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta \leq (c_1 + 1)^2 \int_0^1 e^{\mathring{\mu}+\mathring{\lambda}} \mathring{\rho}(\theta) d\theta t^{-2}, \quad t \geq 1. \quad (3.2)$$

By (3.1) and since $p \leq \rho$,

$$\frac{\partial}{\partial t} e^{\mu-\lambda} = e^{\mu-\lambda} \left[4\pi t e^{2\mu} (p - \rho) + \frac{1 - e^{2\mu}}{t} \right] \leq e^{\mu-\lambda} \left(\frac{1}{t} - \frac{1}{e^{-2\mathring{\mu}} - 1 + t} \right)$$

which implies that

$$e^{\mu(t,\theta) - \lambda(t,\theta)} \leq e^{\mathring{\mu}(\theta) - \mathring{\lambda}(\theta)} \frac{e^{-2\mathring{\mu}(\theta)} t}{e^{-2\mathring{\mu}(\theta)} - 1 + t} \leq e^{|\mathring{\mu}(\theta) - \mathring{\lambda}(\theta)|}, \quad t \geq 1, \quad \theta \in [0, 1]. \quad (3.3)$$

Now we estimate the average of $\mu(t)$, using (3.1), (3.2), (3.3):

$$\begin{aligned} \int_0^1 \mu(t, \theta) d\theta &= \int_0^1 \mathring{\mu}(\theta) d\theta + \int_1^t \int_0^1 \mu_t(s, \theta) d\theta ds \\ &= \int_0^1 \mathring{\mu}(\theta) d\theta + 4\pi \int_1^t s \int_0^1 e^{\mu-\lambda} e^{\mu+\lambda} p d\theta ds + \int_1^t \frac{1}{2s} \int_0^1 (1 - e^{2\mu}) d\theta ds \\ &\leq C + 4\pi \max_{\theta \in [0,1]} e^{|\mathring{\mu}(\theta) - \mathring{\lambda}(\theta)|} (c_1 + 1)^2 \int_0^1 e^{\mathring{\mu}+\mathring{\lambda}} \mathring{\rho} d\theta \ln t \end{aligned}$$

where the constant $C > 0$ depends in a complicated but irrelevant way on the initial data. If the data satisfy the size restriction

$$4\pi \max_{\theta \in [0,1]} e^{|\hat{\mu}(\theta)| - \hat{\lambda}(\theta)} (c_1 + 1)^2 \int_0^1 e^{\hat{\mu} + \hat{\lambda}} \hat{\rho} d\theta \leq \frac{1}{2} \quad (3.4)$$

then the estimate above becomes

$$\int_0^1 \mu(t, \theta) d\theta \leq C + \frac{1}{2} \ln t, \quad t \geq 1.$$

Since

$$\left| \mu(t, \theta) - \int_0^1 \mu(t, \sigma) d\sigma \right| \leq Ct^{-1}, \quad t \geq 1, \quad \theta \in [0, 1],$$

cf. [1, Eqn. 3.14], it follows that

$$\mu(t, \theta) \leq C + \frac{1}{2} \ln t, \quad t \geq 1.$$

Proposition 3.1 *For any solution of the Einstein-Vlasov system with hyperbolic symmetry written in areal coordinates and with initial data as above,*

$$e^{\mu(t, \theta)} \leq C\sqrt{t}, \quad t \geq 1, \quad \theta \in [0, 1],$$

with $C > 0$ depending on the initial data, provided they satisfy the size restriction

$$4\pi \max_{\theta \in [0,1]} e^{|\hat{\mu}(\theta)| - \hat{\lambda}(\theta)} \max\{1, \max_{\theta \in [0,1]} e^{-4\hat{\mu}(\theta)}\} \int_0^1 e^{\hat{\mu} + \hat{\lambda}} \hat{\rho} d\theta \leq \frac{1}{2}.$$

4 Geodesic completeness

Let $] \tau_-, \tau_+[\ni \tau \mapsto (x^\alpha(\tau), p^\beta(\tau))$ be a geodesic whose existence interval is maximally extended and such that $x^0(\tau_0) = t(\tau_0) = 1$ for some $\tau_0 \in] \tau_-, \tau_+[$. We want to show that for timelike and null geodesics which move forward in time, $\tau_+ = +\infty$. Consider first the case of a timelike geodesic, i. e.,

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2, \quad p^0 > 0$$

with $m > 0$. Since $dt/d\tau = p^0 > 0$ the geodesic can be parameterized by the coordinate time t . With respect to coordinate time the geodesic exists on the interval $[1, \infty[$ since on bounded t -intervals the Christoffel symbols are bounded and the right hand sides of the geodesic equations written in coordinate time are linearly bounded in p^1, p^2, p^3 . Along the geodesic we define w and L as above. Then the relation between coordinate time and proper time along the geodesic is given by

$$\frac{dt}{d\tau} = p^0 = e^{-\mu} \sqrt{m^2 + w^2 + L/t^2},$$

and to control this we need to control w as a function of coordinate time. Consider first the case where $w(t) > 0$ for some $t > 1$ and define $t_0 \geq 1$ minimal with the property that $w(s) > 0$ for $s \in]t_0, t]$. We argue similarly to Section 2, making use of the additional estimates which we have established under the size restrictions on the initial data, cf. Props. (2.1) and (3.1):

$$\begin{aligned} \dot{w} &= \frac{4\pi^2}{s} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w} \sqrt{m^2 + w^2 + L/s^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{L}/s^2} \right) f d\tilde{L} d\tilde{w} \\ &\quad + \frac{1 - e^{2\mu}}{2s} w \\ &\leq C \int_0^{P_+(s)} \int_0^{L_0} \frac{\tilde{w}^2(m^2 + w^2 + L/s^2) - w^2(1 + \tilde{w}^2 + \tilde{L}/s^2)}{\tilde{w} \sqrt{m^2 + w^2 + L/s^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{L}/s^2}} f d\tilde{L} d\tilde{w} + \frac{w}{2s} \\ &\leq C \int_0^{P_+(s)} \int_0^{L_0} \frac{\tilde{w}(m^2 + L)}{w} f d\tilde{L} d\tilde{w} + \frac{w}{2s} \leq \frac{Cs}{w} + \frac{w}{2s} \end{aligned}$$

or

$$\frac{d}{ds} w^2 \leq Cs + \frac{1}{s} w^2.$$

This implies that

$$w^2(t) \leq \frac{t}{t_0} w^2(t_0) + Ct(t - t_0).$$

Now either $t_0 > 0$ in which case $w(t_0) = 0$ or $t_0 = 1$. In both cases $w(t) \leq Ct$ where the positive constant C depends on the initial data of the solution of the Einstein-Vlasov system and on the initial data of the particular geodesic. An analogous argument applies if $w(t) < 0$ for some $t > 1$. Thus along the geodesic,

$$\frac{d\tau}{dt} = \frac{e^\mu}{\sqrt{m^2 + w^2 + L/t^2}} \geq \frac{C}{\sqrt{m^2 + Ct^2 + L}}.$$

Since the integral of the right hand side over $[1, \infty[$ diverges, $\tau_+ = +\infty$ as desired.

Now consider a null geodesic which moves forward in time initially, i. e., $m = 0$ and $p^0(\tau_0) > 0$. The quantity L is again conserved. In particular, if $L > 0$ then p^0 remains positive on the maximal existence interval of the geodesic. If $L = 0$ the same is true since otherwise $p^0(\tau) = 0$ and necessarily also $p^1(\tau) = 0$ at some τ which by uniqueness of the solutions of the geodesic equations implies that $p^0 = p^1 = 0$ always, a contradiction. The argument can now be carried out exactly as before, implying that $\tau_+ = +\infty$.

Theorem 4.1 *If the initial data of a solution of the Einstein-Vlasov system with hyperbolic symmetry, written in areal coordinates, satisfy the size restrictions stated in Props. (2.1) and (3.1), then the corresponding spacetime is time-like and null geodesically complete in the expanding direction.*

Final remarks. In [2], conditions on the spacetime metric, the gradient of the lapse, and the extrinsic curvature are given which imply that the spacetime

is timelike and null geodesically complete. These conditions seem stronger than what we needed to establish geodesic completeness in the present note. Among other things, in [2] the lapse function needs to be uniformly bounded from above and away from zero which is more than what our estimates provide.

The control on the w component of the support of f obtained in Section 2 implies the following estimates for the source terms in the field equations:

$$\rho(t, \theta), p(t, \theta), |j(t, \theta)| \leq Ct^{-1}, q(t, \theta) \leq Ct^{-7/2}, t \geq 1, \theta \in [0, 1].$$

If $R_{\alpha\beta\gamma}{}^{\delta}$ denotes the Riemann curvature tensor then the quantity

$$K(t, \theta) := (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})(t, \theta)$$

is known as the Kretschmann scalar. In [3] it was shown that

$$\begin{aligned} K = & 4 \left(e^{-2\lambda} (\mu_{\theta\theta} + \mu_{\theta}(\mu_{\theta} - \lambda_{\theta})) - e^{-2\mu} (\lambda_{tt} + \lambda_t(\lambda_t - \mu_t)) \right)^2 \\ & + \frac{8}{t^2} \left(e^{-4\mu} \lambda_t^2 + e^{-4\mu} \mu_t^2 - 2e^{-2(\lambda+\mu)} \mu_{\theta}^2 \right) + \frac{4}{t^4} (e^{-2\mu} - 1)^2 \end{aligned}$$

and a lower bound for this quantity was established which is positive and blows up as $t \rightarrow 0$. Using the bounds established above one can show that K decays like t^{-2} for $t \rightarrow \infty$.

The question whether the spacetime is geodesically complete for data without a size restriction remains open.

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